

A NOTE ON GRID HOMOLOGY IN LENS SPACES: \mathbb{Z} COEFFICIENTS AND COMPUTATIONS

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ABSTRACT. We present a combinatorial proof for the existence of the sign refined Grid Homology in lens spaces, and a self contained proof that $\partial_{\mathbb{Z}}^2 = 0$. We also present a `SDGE` program that computes $\widehat{GH}(L(p, q), K; \mathbb{Z})$, and provide empirical evidence supporting the absence of torsion in these groups.

1. INTRODUCTION

Ozsváth and Szabó's Heegaard Floer homology [14] is undoubtedly one of the most powerful tools of recent discovery in low dimensional topology. It has far reaching consequences and has been used to solve long standing conjectures (for a survey of some results see [15]). It associates¹ a graded group to a closed and oriented 3-manifold Y , the *Heegaard Floer homology* of Y , by applying a variant of Lagrangian Floer theory in a high dimensional manifold determined by an Heegaard decomposition of Y . Soon after its definition, it was realized independently in [13] and [17] that a knot $K \subset Y$ induces a filtration on the complex whose homology is the Heegaard Floer homology of Y . Furthermore the filtered quasi isomorphism type of this complex is an invariant of the couple (Y, K) , denoted by $HFK(Y, K)$.

The major computational drawback of these theories lies in the differential, which is defined through a count of pseudo-holomorphic disks with appropriate boundary conditions. Nonetheless, a result of Sarkar and Wang [19] ensures that after a choice of a suitable doubly pointed Heegaard diagram \mathcal{H} for (Y, K) , the differential can be computed directly from the combinatorics of \mathcal{H} . If moreover Y is a rational homology 3-sphere such that $g(Y) = 1$ (*i.e.* $Y = S^3$ or $L(p, q)$), the whole complex $HFK(Y, K)$ admits a neat combinatorial definition, known as *grid homology*.

Grid homology in S^3 was pioneered by Manolescu, Ozsváth and Sarkar in [8], and for lens spaces by Baker, Hedden and Grigsby in [1]; as the name suggests both the ambient manifold and the knot are encoded in a grid, from which complex and differential for the grid homology can be extracted by simple combinatorial computations.

¹There are actually many different variants of the theory, which we ignore presently; in the following Sections we will define some variants which will be relevant to our discussion.

After establishing the necessary background, we will present in Section 2 the definition of grid homology in lens spaces as given in [1].

Here, we produce a purely combinatorial proof that $\partial^2 = 0$.

Section 3 is devoted to prove existence and uniqueness for *sign assignments* of grid diagrams:

Theorem 1.1. *Sign assignments exist on all grids representing a knot $K \subset L(p, q)$, and can be described combinatorially. Moreover the sign refined grid homology does not depend on the choice of a sign assignment.*

The sign refinement of the theory is carried on using a group theoretic reformulation of sign assignments due to Gallais ([6]).

Finally in Section 4 we present some computations and examples, together with a description of the program used to make them. This program is available on my homepage:

<http://poisson.dm.unipi.it/~celoria/#programs>

With this tool we are able to show:

Proposition 1.2. *The sign refined grid homology of knots with small parameters is torsion free.*

This result provides empirical evidence for the absence of torsion in the knot Floer homology of knots in lens spaces. Analogous results for knots in the three sphere have been found by Droz in [5].

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2. GRID HOMOLOGY IN LENS SPACES

2.1. Representing knots with grids. In what follows p, q will always be two coprime integers, and the lens space $L(p, q)$ is the (closed, oriented) 3-manifold obtained by $-\frac{p}{q}$ surgery on the unknot $\bigcirc \subset S^3$; to avoid confusion, a knot in a 3-manifold Y will be usually denoted by (Y, K) .

Definition 2.1. *Consider a $n \times pn$ grid in \mathbb{R}^2 , consisting of the segments $\tilde{\alpha}_i = (tnp, i)$ and $\tilde{\beta}_j = (j, tnp)$ with $i \in \{0, \dots, n\}$, $j \in \{0, \dots, np\}$ and $t \in [0, 1]$. A twisted grid diagram for $L(p, q)$ is the grid on the torus given by identifying $\tilde{\beta}_0$ to $\tilde{\beta}_{pn}$, and then $\tilde{\alpha}_0$ to $\tilde{\alpha}_n$ according to a twist depending on q (see Fig. 2.1):*

$$\alpha_n \ni (s, n) \sim (s - qn \pmod{pn}, 0) \in \alpha_0$$

Here $s \in [0, pn]$; the condition $(p, q) = 1$ guarantees that after the identifications the planar grid becomes a toroidal grid.

Call $\alpha = \{\alpha_i\}$ and $\beta = \{\beta_i\}$ $i \in \{1, \dots, n\}$ the n horizontal (resp. vertical) circles obtained after the identifications in the grid.

We can encode a link L in $L(p, q)$ by placing a suitable version of the \mathbb{X} 's and \mathbb{O} 's for grid diagrams in S^3 : call $\mathbb{X} = \{\mathbb{X}_i\}$ and $\mathbb{O} = \{\mathbb{O}_i\}$, $i = 1, \dots, n$ two sets of markings. Put each one of them in the little squares² of $G \setminus \alpha \cup \beta$ in such a way that each column³ and row contains exactly one element of \mathbb{X} and one of \mathbb{O} , and each square contains at most one marking.

Now join with a segment each \mathbb{X} to the \mathbb{O} which lies on the same row, and each \mathbb{O} to the \mathbb{X} which lies on the same column (keeping in mind the twisted identification); with this convention we can encode an orientation⁴ for the link. To get an honest link remove self intersections by resolving each crossing as an overcrossing of the vertical segments over the horizontal ones (as in Figure 2.2).

The grid together with the markings is a multipointed Heegaard diagram for $(L(p, q), K)$ and

$$|\alpha_i \cap \beta_j| = p \quad \forall i, j \in \{1, \dots, n\}$$

Remark 2.2. There are two possible ways to connect each \mathbb{X}_i to the corresponding \mathbb{O} marking on the same row/column, but the isotopy class of the resulting link does not depend upon the possible choices. Indeed the two choices for each row/column are topologically related by a slide on a meridional disk of the Heegaard decomposition of $L(p, q)$, hence describe isotopic links.

The integers n, p and q will be called the *parameters* of the grid diagram G ; the p ($n \times n$) squares obtained by cutting the torus along α_1 and β_1 (in the planar representation of the grid) are called *boxes*. We will often deliberately forget the distinction between planar and toroidal grids, according to the motto “draw on a plane, think on a torus”.

It is worth to point out that the case in which $p = 1$ and $q = 0$ gives as expected a usual grid diagram for a link in S^3 .

Remark 2.3. Exchanging the role of the markings in a grid representing a knot K produces a grid diagram for the same link with the opposite orientation on each component.

Proposition 2.4 ([2],[1]). *Every link in $L(p, q)$ can be represented by a grid diagram; two different grid representations of a link differ by a*

²For concreteness think at the markings as having half integral coordinates in the planar grid.

³Beware! In a twisted toroidal grid a column “wraps around” a row p times.

⁴Note that this convention is the opposite of the one used in [16], but agrees with the one of [1]; see also Remark 2.3.

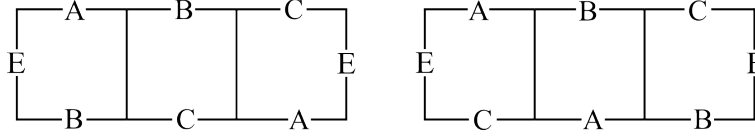


FIGURE 2.1. Top-bottom identifications for a 3 dimensional grid for $L(3, 1)$ (left) and $L(3, 2)$ (right).

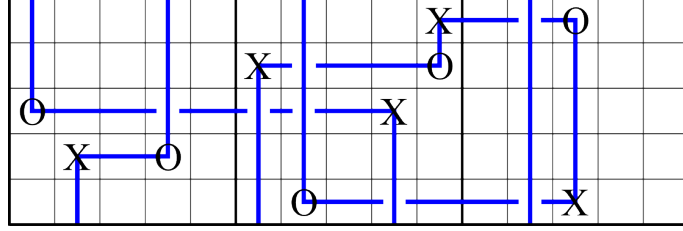


FIGURE 2.2. The link obtained by joining \mathbb{X} 's and \mathbb{O} 's in a grid for $L(3, 2)$ of grid dimension 5.

finite number of grid moves analogous to Cromwell's for grid diagrams in S^3 :

- *Translations: these are just vertical and horizontal integer shifts of the grid (keeping the identifications in mind).*
- *(non-interleaving) Commutations: if two adjacent row/columns c_1 and c_2 are such that the markings of c_1 are contained in a connected component of c_2 with the two squares containing the markings removed, then they can be exchanged.*
- *(de)Stabilizations: these are the only moves that change the dimension of the grid. There are 8 types of stabilization, as shown in Figure 2.3. Destabilizations are just the inverse moves.*

Remark 2.5. The homology class of a knot $K \subset L(p, q)$ can be read directly from the grid; we just need to keep track of the signed number of intersections of the knot with a meridian of the torus. With the orientation conventions we have established (so that vertical arcs connect \mathbb{O} 's to \mathbb{X} 's):

$$H_1(L(p, q); \mathbb{Z}) \ni [K] = \#\{\alpha_1 \cap K\} \pmod{p}$$

Remark 2.6. If G is a grid of parameters (n, p, q) , we call n the dimension (or grid number) of G . The same term will also be used when referring to the isotopy class of a knot $(L(p, q), K)$; in this case we mean the quantity

$$\min\{n \mid G \text{ is a grid with parameters } (n, p, q) \text{ representing } K\}$$

2.2. Generators of the complex. In the following we are going to define two different versions (sometimes also known as *flavors*) of the grid homology for knots in lens spaces. They can all be defined by slight variations in the complex, the ground ring or the differential we

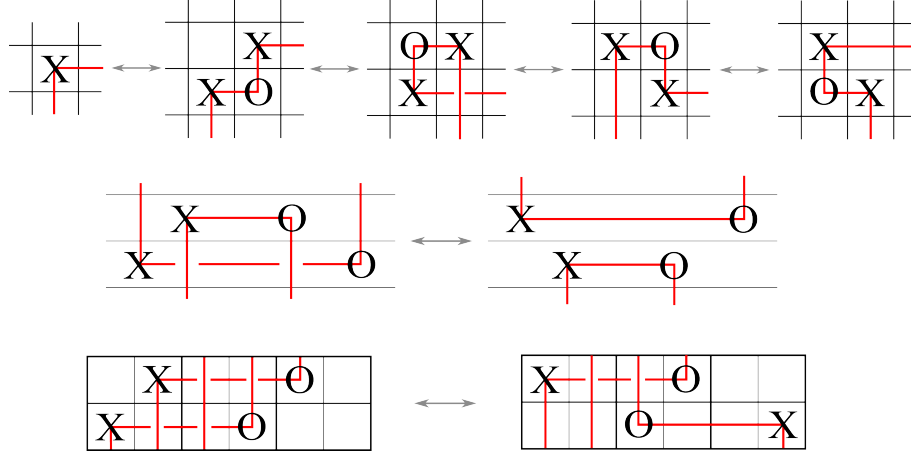


FIGURE 2.3. Some examples of grid moves; in the top there are four different kinds of stabilizations (there are other four where the roles of the markings are exchanged). In the middle a row commutation. On the lower part an example of vertical translation in $L(3, 1)$.

will introduce below. For clarity we are going to restrict ourselves to $\mathbb{F} = \mathbb{Z}_2$ coefficients until the next Section, and to knots throughout the paper.

Definition 2.7. *Given a grid G of dimension n representing a knot $K \subset L(p, q)$, the generating set for G is the set $S(G)$ comprising all bijections between α and β curves. This corresponds to choosing n points in $\alpha \cap \beta$ such that there is exactly one on each α and β curve. There is a bijection*

$$S(G) \longleftrightarrow \mathfrak{S}_n \times (\mathbb{Z}_p)^n$$

which can be described as follows: since we fixed a cyclic labelling of the α and β curves it makes sense to speak of the m -th intersection between two curves, with $0 \leq m \leq p-1$; so if the l -th component of a generator lies on the m -th intersection of α_l and β_j then the permutation $\sigma \in \mathfrak{S}_n$ associated will be such that $\sigma(l) = j$ and the l -th component of $(\mathbb{Z}_p)^n$ will be m .

If $x \in S(G)$, we can thus write $x = (\sigma_x, (x_1^p, \dots, x_n^p))$; we will refer to σ_x as the permutation component of the generator, and to (x_1^p, \dots, x_n^p) as its p -coordinates.

$S(G)$ can be endowed with a $(\mathbb{Q}, \mathbb{Q}, \mathbb{Z}_p)$ -valued grading. The first two degrees are known as *Maslov* and *Alexander* degrees. The last one is the $Spin^c$ degree; since it is preserved by the differential (Proposition 2.13), it will provide a splitting of the complex in to p direct summands. All these degree are going to be defined in a purely combinatorial way.

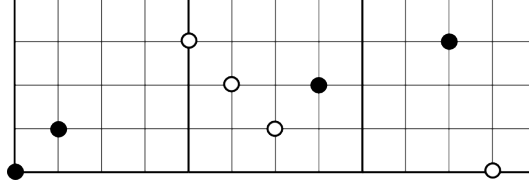


FIGURE 2.4. Under the bijection described before the white generator corresponds to $((14)(23), (2, 1, 1, 1))$, and the black to $((34), (0, 0, 1, 2)) \in \mathfrak{S}_4 \times (\mathbb{Z}_3)^4$.

Definition 2.8. Let A and B denote two finite sets of points in \mathbb{R}^2 ; call $\mathcal{I}(A, B)$ the number of pairs

$$((a_1, a_2), (b_1, b_2)) \subset A \times B$$

such that $a_i < b_i$ for $i = 1, 2$.

Denote by $X(p, n)$ (respectively $Y(p, n)$) the set of n -tuples (respectively pn -tuples) of points contained in the $n \times pn$ (respectively $pn \times pn$) rectangle in \mathbb{R}^2 whose bottom vertices are $(0, 0)$ and $(pn, 0)$; then define

$$C_{p,q} : X(p, n) \longrightarrow Y(p, n)$$

the function which sends an n -tuple $\{(c_i, b_i)\}_{i=1, \dots, n}$ to the pn -tuple

$$\{(c_i + nqk \pmod{np}, b_i + nk)\}_{\substack{i=1, \dots, n \\ k=0, \dots, p-1}}$$

In order to avoid notational overloads, we are going to write \tilde{x} instead of $C_{p,q}(x)$.

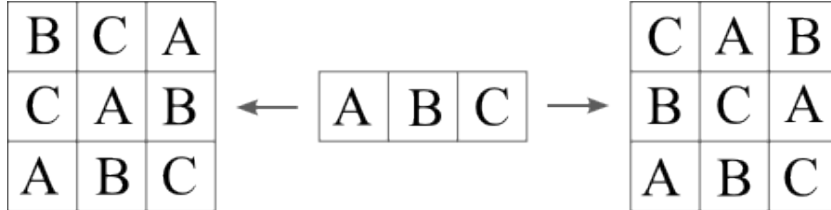


FIGURE 2.5. A representation of the action of $C_{p,q}$ for $(p, q) = (3, 1)$ and $(3, 2)$ (on the left and right respectively).

We can then define the Maslov degree:

(2.1)

$$M(x) = \frac{1}{p} \left[\mathcal{I}(\tilde{x}, \tilde{x}) - \mathcal{I}(\tilde{x}, \tilde{\mathcal{O}}) - \mathcal{I}(\tilde{\mathcal{O}}, \tilde{x}) + \mathcal{I}(\tilde{\mathcal{O}}, \tilde{\mathcal{O}}) \right] + d(p, q, q-1) + 1$$

$d(p, q, q-1)$ is a rational number known as the *correction term* of $L(p, q)$ associated to the $(q-1)$ -th $Spin^c$ structure; following [12], it can be computed recursively as follows⁵:

⁵A user-friendly online calculator for these correction terms can be found at http://poisson.dm.unipi.it/~celoria/correction_tems.html

- $d(1, 0, 0) = 0$
- $d(p, q, i) = \left(\frac{pq - (2i+1-p-q)^2}{4pq} \right) + d(q, r, j)$ where r and j denote the reduction of p and $i \pmod{q}$.

Similarly the Alexander grading can be defined as:

$$(2.2) \quad A(x) = \frac{1}{2p} \left[\mathcal{I}(\tilde{\mathbb{O}}, \tilde{\mathbb{O}}) - \mathcal{I}(\tilde{\mathbb{X}}, \tilde{\mathbb{X}}) + 2\mathcal{I}(\tilde{\mathbb{X}}, \tilde{x}) - 2\mathcal{I}(\tilde{\mathbb{O}}, \tilde{x}) \right] + \frac{1-n}{2}$$

Remark 2.9. By slightly modifying the differential, A can be demoted to a filtration on the complex, rather than a degree. The complexes we are going to consider should be thought as the graded objects associated to this filtration. Note that 2.2 is not the standard formula used to define A ; here we are using the fact (see [3]) that in a grid of dimension n for a knot in S^3

$$\mathcal{I}(x, J) - \mathcal{I}(J, x) = n$$

with $J = \mathbb{O}$ or \mathbb{X} , and x is a generator.

Now call $(a_1^{\mathbb{O}}, \dots, a_n^{\mathbb{O}})$ the p -coordinates of the generator whose components are in the lower left vertex of the squares which contain an \mathbb{O} marking. The $Spin^c$ degree of $x = (\sigma_x, (a_1, \dots, a_n)) \in S(G)$ is defined⁶ as:

$$S : \mathfrak{S}_n \times (\mathbb{Z}_p)^n \longrightarrow \mathbb{Z}_p$$

$$(2.3) \quad S(x) = q - 1 + \sum_{i=1}^n (a_i - a_i^{\mathbb{O}}) \pmod{p}$$

The Alexander grading depends on the placement of all the markings, while M and S only on the position of the \mathbb{O} 's.

Let $R = \mathbb{F}[V_1, \dots, V_n]$ denote the ring of n -variable polynomials with \mathbb{F} coefficients, and $\hat{R} = R / \{V_1 = 0\}$. These V variables⁷ are graded endomorphisms of the complex; their function is to “keep track” of the \mathbb{O} markings in the differential.

We can now define at least the underlying module structure of the complexes we are going to use in the following:

Definition 2.10. *The minus complex $GC^-(G)$ is the free R module generated over $S(G)$. The hat complex $\widehat{GC}(G)$ is the free \hat{R} -module generated over $S(G)$. Extend the gradings to the whole module by setting the behavior of the action for the variables in the ground ring:*

$$\begin{aligned} A(V \cdot x) &= A(x) - 1 \\ M(V \cdot x) &= M(x) - 2 \\ S(V \cdot x) &= S(x) \end{aligned}$$

⁶We are implicitly using an identifications between $Spin^c(L(p, q))$ and \mathbb{Z}_p (cfr. [12, Sec. 4.1]).

⁷We adopt here the convention of [16], in order to stress the difference between the endomorphisms on the complex (the V_i 's) and the induced map on homology, which will be denoted by U .

where V is any of the V_i 's.

Example 2.11. In this example we are going to exhibit the generating set of the grid G on the left of Figure 2.6, in the 0-th $Spin^c$ structure, which we are going to denote by $S(G, 0)$. $S(G, 0)$ is composed by 4 elements:

$$a = \mathbb{F}_{[-\frac{1}{4}, -\frac{1}{4}]}, b = \mathbb{F}_{[-\frac{1}{4}, -\frac{1}{4}]}, c = \mathbb{F}_{[\frac{3}{4}, -\frac{1}{4}]}, d = \mathbb{F}_{[-\frac{5}{4}, -\frac{5}{4}]}$$

The notation $\mathbb{F}_{[a,b]}$ denotes a generator having (a, b) bidegree.

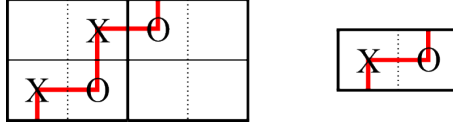


FIGURE 2.6. A grid for the knot considered, and the same grid after a destabilization. Confront this computations with Remark 2.21.

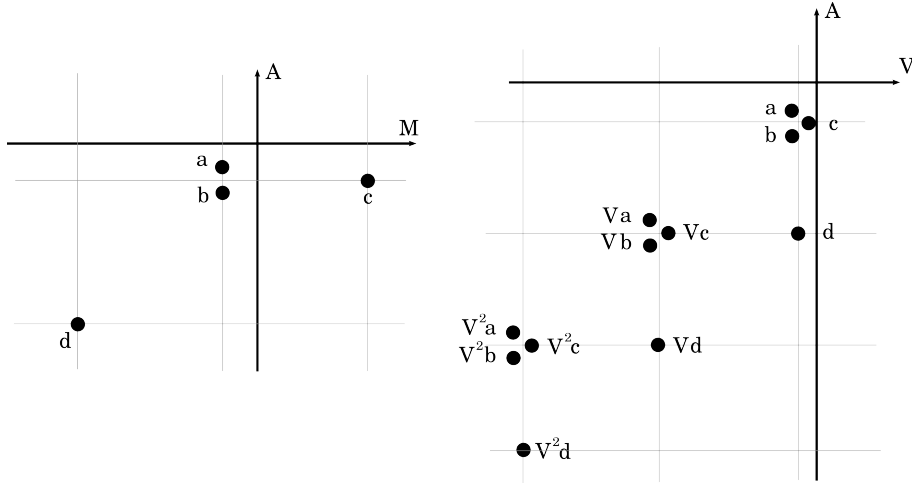


FIGURE 2.7. (left) The generating set $S(G, 0)$, with the bidegree (M, A) on the axes.

(right) The underlying modules for the complexes $\widehat{GC}(G, 0) = GC^-(G, 0)$ with axes labeled by powers of the V variables and Alexander degree. The dots represent generators over \mathbb{F} .

2.3. The differential. As already mentioned in the introduction, grid homology hinges upon Sarkar and Wang result of [19]; in their terminology, (twisted) grid diagrams are *nice* (multipointed, genus 1) Heegaard diagram for $L(p, q)$, so the differential of CFK can be computed

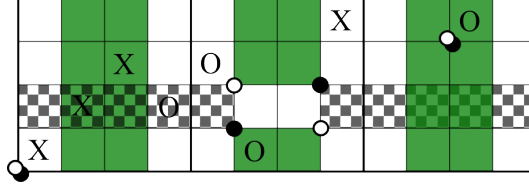


FIGURE 2.8. Two oriented rectangles connecting x (white) to y (black) in a grid of parameters $(4,3,1)$. Only the horizontal one (colored with a checkerboard pattern) is empty.

combinatorially. In this contest the holomorphic disks of Knot Floer homology take the milder form of embedded rectangles on the grid.

Consider two generators x and $y \in S(G)$ having the same $Spin^c$ degree; if the permutations associated to x and y differ by a transposition, then the two components where the generators differ are the vertices of four immersed rectangles r_1, \dots, r_4 in the grid; the sides of the r_i 's are alternately arcs on the α and β curves. We can fix an orientation for such a rectangle r , by prescribing that r goes from x to y if its lower left and upper right corners are on x components. This cuts the number of rectangles connecting two generators that differ by a transposition to 2.

Definition 2.12. Given a grid G , and x, y of $S(G)$, call $Rect(x, y)$ the set of oriented rectangles connecting x to y ; we will denote by

$$Rect(G) = \bigcup_{x, y \in S(G)} Rect(x, y)$$

the set of all oriented rectangles between generators in G . Similarly $Rect^\circ(G)$ is going to be the set of empty rectangles, that is the rectangles $r \in Rect(x, y)$ for which $Int(r) \cap x = \emptyset$.

Note that by assumption if $r \in Rect(x, y)$ is empty, then it does not contain any point of y either, so in particular it is embedded in the torus composed by the grid.

If $x, y \in S(G)$, then $|Rect(x, y)| \in \{0, 2\}$, and it can be non zero only for generators in the same $Spin^c$ degree which differ by a single transposition. On the other hand with the same hypothesis on the generators, $|Rect^\circ(x, y)| \in \{0, 1, 2\}$. If $r_1 \in Rect(x, y)$ and $r_2 \in Rect(y, z)$ we can consider their concatenation $r_1 * r_2$, which we call a *polygon* connecting x to z through y . We are going to denote by $Poly(x, z)$ the set of polygons connecting x to z , and by $Poly^\circ(x, z)$ the empty ones. If P is an empty rectangle or polygon, denote by $O_i(P)$ the number of times that the i -th \odot marking appears in P . In a grid diagram for knots in S^3 , $O_i(P) \in \{0, 1\}$, but if P is a polygon in a twisted grid, then $O_i(P) \in \{0, 1, 2\}$ (see Fig. 2.12).

The differential is just going to be a count of empty rectangles, satisfying some additional constraints according to the flavor chosen. For the two flavors of grid homology considered here⁸ we keep track of the \mathbb{O} markings contained in the rectangles, by multiplying with the corresponding variable V_i :

$$(2.4) \quad \partial(x) = \sum_{y \in S(G)} \sum_{\substack{r \in \text{Rect}^\circ(x,y) \\ r \cap \mathbb{X} = \emptyset}} \left(\prod_{i=1}^n V_i^{O_i(r)} \right) y$$

Proposition 2.13. *Given a grid diagram G of parameters (n, p, q) , the modules $GC^-(G)$ and $\widehat{GC}(G)$ endowed with the endomorphism ∂ are chain complexes, that is $\partial^2 = 0$ in both cases. Moreover ∂ acts on the trigrading as follows:*

- (1) $S(\partial(x)) = S(x)$
- (2) $M(\partial(x)) = M(x) - 1$
- (3) $A(\partial(x)) = A(x)$

Remark 2.14. This Proposition is implicit in [1], and it can be seen as a direct consequence of Theorem 1.1 therein; however some of the considerations in this proof will be useful in the following section. Moreover this proof will rely only on combinatorial considerations, showing that the result can be obtained without any reference to the holomorphic theory of [13] and [17].

Proof. We begin by examining the behavior of the degrees under the differential; condition (1) is easy to prove: by equation 2.3 the only relevant part of a generator x for the computation of $S(x)$ is given by its p -coordinates. If y appears in the differential of x , call a_i^x, a_i^y, a_j^x and a_j^y the p -coordinates where they differ. If $a_i^x = k$ and $a_j^x = l$, with $k, l \in \mathbb{Z}_p$, then (since they are connected by a rectangle) $a_i^y = k + t$ and $a_j^y = l - t$ modulo p for some $t \in \{0, \dots, p-1\}$; so $S(x) = S(y)$.

If x and y are generators in G connected by an empty rectangle r , then

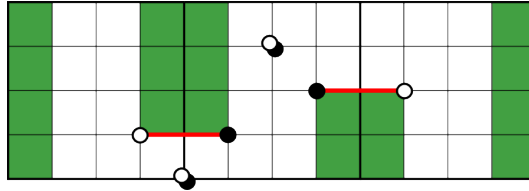


FIGURE 2.9. The non equal p -coordinates of the generators compensate with one another.

their lifts \tilde{x} and \tilde{y} will differ in $2p$ positions, according to the pattern suggested in Fig. 2.10. This implies that the corresponding \mathcal{I} function

⁸Keep in mind that for \widehat{GC} we set $V_1 = 0$.

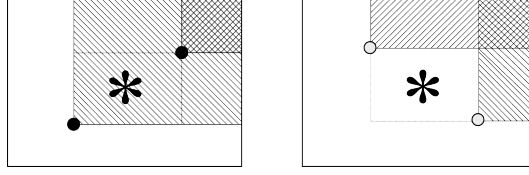


FIGURE 2.10. The difference between the functions $\mathcal{I}(x, *)$ and $\mathcal{I}(y, *)$ for two generators whose permutations differ by a transposition.

will change accordingly:

$$\begin{aligned}\mathcal{I}(\tilde{x}, \tilde{x}) &= \mathcal{I}(\tilde{y}, \tilde{y}) + p \\ \mathcal{I}(\tilde{x}, \tilde{\mathbb{O}}) &= \mathcal{I}(\tilde{y}, \tilde{\mathbb{O}}) + p \sum_{i=1}^n O_i(r) \\ \mathcal{I}(\tilde{\mathbb{O}}, \tilde{x}) &= \mathcal{I}(\tilde{\mathbb{O}}, \tilde{y}) + p \sum_{i=1}^n O_i(r)\end{aligned}$$

And the same result holds with \mathbb{X} markings instead of \mathbb{O} 's. Then from equation 2.4 we get for (2) and (3) respectively:

$$\begin{aligned}M(\partial(x)) &= \sum_{y \in S(G)} \sum_{\substack{r \in \text{Rect}^\circ(x, y) \\ r \cap \mathbb{X} = \emptyset}} \left(\sum_{i=1}^n -2O_i(r) \right) M(y) \\ A(\partial(x)) &= \sum_{y \in S(G)} \sum_{\substack{r \in \text{Rect}^\circ(x, y) \\ r \cap \mathbb{X} = \emptyset}} \left(\sum_{i=1}^n -O_i(r) \right) A(y)\end{aligned}$$

A substitution using equations 2.1 and 2.2 defining the Maslov and Alexander degrees yields (2) and (3).

We are left to show that $\partial^2 = 0$; we thus need to study the possible decompositions in rectangles of polygons connecting two generators.

We will prove the result for the minus flavored complex, since the analogous result for the hat version follows immediately. From 2.4 we can compute

$$(2.5) \quad \partial^2(x) = \sum_{z \in S(G)} \sum_{\substack{\psi \in \text{Poly}^\circ(x, z) \\ \psi \cap \mathbb{X} = \emptyset}} N(\psi) \left(\prod_{i=1}^n V_i^{O_i(\psi)} \right) z$$

where ψ is a polygon connecting x to z , and $N(\psi)$ is the number of possible ways of writing ψ as the composition of two empty rectangles $r_1 * r_2$, with $r_1 \in \text{Rect}^\circ(x, y)$ and $r_2 \in \text{Rect}^\circ(y, z)$ for some $y \in S(G)$. Note that a polygon P connecting two generators is empty if and only if so are the rectangles P is made of. In order to complete the proof we need to show that $N(\psi) \equiv 0 \pmod{2}$, *i.e.* there is an even number of

ways⁹ (in fact 2) to decompose into rectangles a fixed ψ that appears in the squared differential ∂^2 . We can also take advantage of the proof in [16, Lemma 4.4.6] to reduce the number of cases to examine; as a matter of fact, if a polygon ψ does not cross one of the α curves, we can cut the torus open along it, and think of the polygon as living in a portion of an $np \times np$ grid for S^3 . Thus we only need to worry about polygons that intersect all the α circles.

There are then four possibilities to be considered a priori, according to the quantity $M = |x \setminus (x \cap z)| \in \{0, \dots, 4\}$, as schematically shown in Figure 2.11. If $M = 0$, that is $x = z$, the only possible polygons are

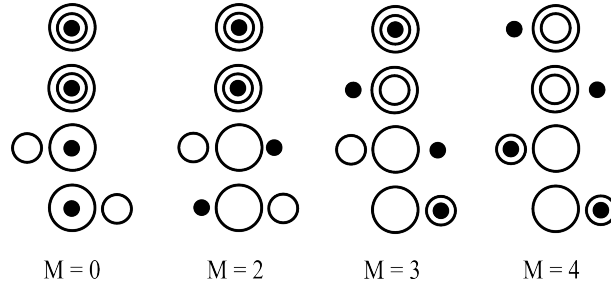


FIGURE 2.11. A representation of the possibilities for M (in a grid of dimension 4). The circles correspond (from big to small) to the components of generators x, y and z . Two circles are concentric if the corresponding generators coincide.

thin rectangles, called α and β degenerations. These are strips of respectively height or width 1 (otherwise they would not be empty). We are not concerned with these strips, since each of them contains exactly one \mathbb{X} marking, hence they do not contribute to the differential.

As an aside we note here that there is only one way to decompose such a strip into two rectangles (one starting from x , and one arriving to it). The case $M = 1$ can be dismissed too, since rectangles only connect generators which differ in exactly two points¹⁰.

If $M = 4$, that is the corners of the two rectangles are all distinct, we can apply the same approach of [16]; there are two ways of counting them, as shown in Fig. 2.12. Basically the two decompositions correspond to taking the two rectangles in either order. We remark that one rectangle might wrap around the other, but the number of decompositions does not depend on this wrapping.

The case $M = 2$ needs a bit more care since it has no S^3 counterpart

⁹This is not true for the *filtered* versions of these complexes. Nonetheless the polygons that cannot be split in two different ways cancel each other out nicely in that case too.

¹⁰And a product of two nontrivial and different transpositions is never a transposition.

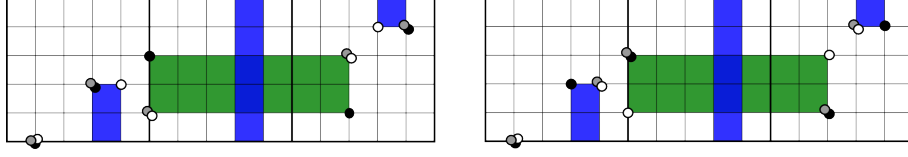


FIGURE 2.12. When $M = 4$ we can consider the two rectangles (from white to black) in either order, by choosing a suitable intermediate generator y (gray).

(see [16, Ch. 4]). In this case the two rectangles must share part of 2 edges. There are two possibilities:

- (1) the rectangle starting from x does not cross all the α curves. Up to vertical/horizontal translations it can be placed in such a way that it does not intersect the boundary of the planar grid.
- (2) the rectangle starting from x intersects all the α curves at least once.

Either way, the second rectangle joining the intermediate generator (y or w in the notation above) to z must end and start on the same α curves of the first rectangle; the configurations in both cases are shown Fig 2.13, together with their decompositions.

Lastly, if $M = 3$ we can again distinguish two possibilities as in the previous case; the combinatorially inequivalent configurations are shown in Figure 2.14, again with their two decompositions. \square

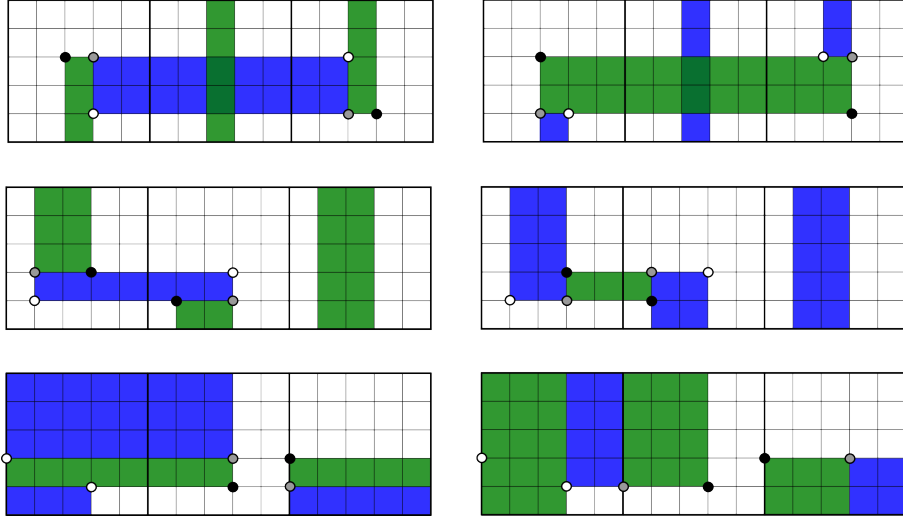


FIGURE 2.13. Relevant combinatorial possibilities for $M = 2$ on a grid for $L(3, 1)$. On each row the two possible decompositions are shown. Again we adopt the convention $x, y, z = \text{white/gray/black dots}$, showing only the relevant components.

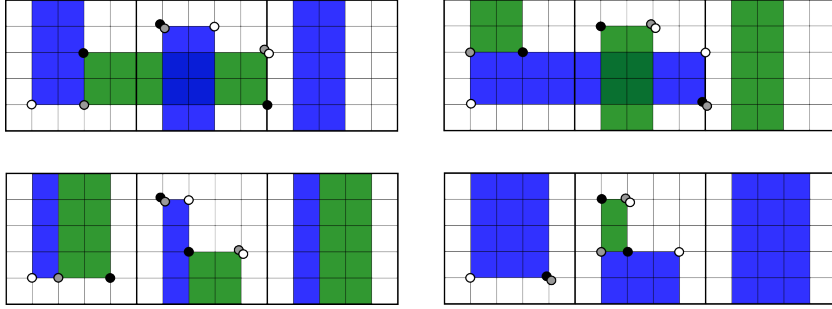


FIGURE 2.14. Some configurations for the $M = 3$ case. The complete combinatorial classification up to wrapping is presented in Figure 3.5.

Example 2.15. We continue here the computations of example 2.11: we can now complete the picture by adding the differentials and computing the various homologies. We have:

$$\begin{aligned}\partial(a) &= \partial(b) = 0 \\ \partial(c) &= a + b \\ \partial(d) &= V_1a + V_2b\end{aligned}$$

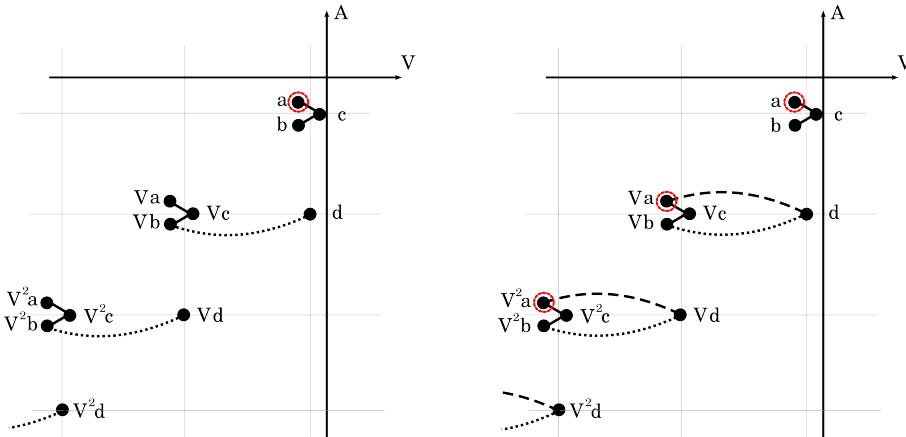


FIGURE 2.15. On the left the complex $\widehat{GC}(G, 0)$ and on the right the complex $GC^-(G, 0)$; the dotted line corresponds to multiplication by V_2 , and the hatched one to multiplication by V_1 . Non trivial elements in homology are circled.

It is then an easy task to compute the grid homologies in the two flavors:

$$\begin{aligned}\widehat{GH}(G, 0) &= \mathbb{F}\langle a \rangle \cong \mathbb{F}_{[-\frac{1}{4}, -\frac{1}{4}]} \\ GH^-(G, 0) &= \mathbb{F}[V_1]\langle a \rangle \cong \mathbb{F}[U]_{[-\frac{1}{4}, -\frac{1}{4}]}\end{aligned}$$

2.4. The homologies. From the definitions given up to now it might seem strange that the homology of such a complex could be an invariant of the smooth isotopy type of a knot, since even the ground ring depends on the dimension of a grid representing it; Theorem 2.17 below ensures however that GH^- and \widehat{GH} are quasi-isomorphic to a finitely generated $\mathbb{F}[U]$ and \mathbb{F} modules respectively. The algebraic reason behind this is the content of the following Proposition:

Proposition 2.16. *Let G be a grid of parameters (n, p, q) for a knot K . Then the action of multiplication by V_i on the complex $GC^-(G)$ is quasi isomorphic to multiplication by V_j .*

Proof. See [16, Ch. 4]. □

Theorem 2.17 ([1]). *The homologies*

$$GH^-(G) = H_*(GC^-(G), \partial)$$

and

$$\widehat{GH}(G) = H_*(\widehat{GC}(G), \partial)$$

regarded as $(\mathbb{Q}, \mathbb{Q}, \mathbb{Z}_p)$ graded modules over the appropriate ring are invariants of the knot $(L(p, q), K)$. Moreover $(GC^-(G), \partial)$ is quasi isomorphic to a finitely generated $\mathbb{F}[U]$ module, where U acts as any of the V_i 's, and (\widehat{GC}, ∂) is quasi isomorphic to a finitely generated \mathbb{F} module.

Proof. Rather than adapting the analogous of the combinatorial proof in [16] to $L(p, q)$, we appeal to the main result of [1]. □

Due to this Theorem we will sometimes make the notational abuse of writing $\widehat{GH}(L(p, q), K)$ instead of $\widehat{GH}(G)$, G being a grid of parameters (n, p, q) representing K .

Remark 2.18. Since the differential preserves the decomposition of the complex in $Spin^c$ structures (Prop 2.13), we can write

$$GH^-(L(p, q), K) = \bigoplus_{s \in \mathbb{Z}_p} GH^-(L(p, q), K, s)$$

$$\widehat{GH}(L(p, q), K) = \bigoplus_{s \in \mathbb{Z}_p} \widehat{GH}(L(p, q), K, s)$$

and according to Prop. 2.13 the endomorphism U induced in homology by any of the V_i acts as

$$U : GH_m^-(L(p, q), K, s, a) \longrightarrow GH_{m-2}^-(L(p, q), K, s, a - 1)$$

where $GH_m^-(L(p, q), K, s, a)$ is the homology in trigrading $(m, a, s) \in (\mathbb{Q}, \mathbb{Q}, \mathbb{Z}_p)$.

We can finally state the main result of [1]:

Theorem 2.19 ([1]). *Let G be a grid for a knot $K \subset L(p, q)$. There is a graded isomorphism of $\mathbb{F}[U]$ and respectively \mathbb{F} trigraded modules:*

$$\begin{aligned} HFK^-(L(p, q), K) &\cong GH^-(G) \\ \widehat{HFK}(L(p, q), K) &\cong \widehat{GH}(G) \end{aligned}$$

Remark 2.20. Knot Floer homology is known (see *e.g.* [13]) to satisfy a formula¹¹ for the connected sum of two knots in rational homology 3-spheres; if $(Y, K) = (Y_0, K_0) \# (Y_1, K_1)$, then

(2.6)

$$HFK^-(Y, K, i) \cong \bigoplus_{i_0+i_1=i} HFK^-(Y_0, K_0, i_0) \otimes_{\mathbb{F}[U]} HFK^-(Y_1, K_1, i_1)$$

In each connected 3-manifold Y the isotopy class of the homologically trivial unknot \bigcirc is unique (since it bounds an embedded disk and manifolds are homogeneous); thus we can think of a *local* knot K , *i.e.* a knot contained in a 3-ball inside Y as the connected sum

$$(Y, K) = (Y, \bigcirc) \# (S^3, K')$$

for some knot K' in S^3 . It is a straightforward computation to show that the grid homology of the unknot $\bigcirc \subset L(p, q)$ is:

$$\begin{aligned} GH^-(L(p, q), \bigcirc) &= \bigoplus_{s \in \mathbb{Z}_p} \mathbb{F}[U]_{[d(p, q, s), 0]} \\ \widehat{GH}(L(p, q), \bigcirc) &= \bigoplus_{s \in \mathbb{Z}_p} \mathbb{F}_{[d(p, q, s), 0]} \end{aligned}$$

So by 2.6

$$GH^-(L(p, q), K) = \bigoplus_{s \in \mathbb{Z}_p} GH^-(S^3, K')_{[d(p, q, s), *]}$$

In other terms the grid homology of a local knot is completely determined by the homology of the same knot viewed as living in S^3 (and in particular its Alexander degrees are integers).

Remark 2.21. A *simple knot* in $L(p, q)$ is a knot admitting a grid of dimension 1, (see also [7] and [18]). It is easy to show that in each lens space $L(p, q)$ there is exactly one simple knot in each homology class; for $m \in H_1(L(p, q); \mathbb{Z})$ denote this knot by $T_m^{p, q}$.

If G is the dimension 1 grid representing it, then $|S(G)| = p$, and there is exactly one generator in each $Spin^c$ degree. There cannot be any differentials (since ∂ preserves the $Spin^c$ degree), so the homologies of $T_m^{p, q}$ are:

$$GH^-(T_m^{p, q}) \cong \bigoplus_{s \in \mathbb{Z}_p} \mathbb{F}[U]_{[d(p, q, s), A(x_s)]}$$

¹¹We do not specify here the various conventions involved for $Spin^c$ structures, since in what follows we will only deal with $Y = S^3, L(p, q)$.

$$\widehat{GH}(T_m^{p,q}) \cong \bigoplus_{s \in \mathbb{Z}_p} \mathbb{F}_{[d(p,q,s), A(x_s)]}$$

where $A(x_s)$ is the Alexander degree of the unique generators in degree s . As in [1] we say that these knots are Floer simple (or U -knot in the terminology of [12]), meaning that the rank of the grid homology (over the appropriate ground ring) is exactly one in each $Spin^c$ degree. Explicit computations for $GH^-(T_m^{p,q})$ and related invariants will be detailed in an upcoming paper. A recursive formula can be instead found in [18].

3. LIFT TO \mathbb{Z} COEFFICIENTS

The complexes we have used until now were defined to work with \mathbb{F} as base ring; in particular the proof of Proposition 2.13 relied on the parity of polygon decompositions to ensure that (GC^-, ∂) is in fact a chain complex. This section is devoted to a combinatorial extension of the previous construction with \mathbb{Z} coefficients. This was first done in the combinatorial setting for S^3 in [9] (see also [11]).

We will adopt the group theoretic approach first developed in [6] to define a sign function on rectangles, whose properties are precisely tuned to have $\partial^2 = 0$. One might ask how the theory changes under such a change of coefficients; at the time of writing there is no example of knot in S^3 whose knot Floer homology with \mathbb{Z} coefficients exhibits torsion (see Problem 17.2.9 of [16]). Even in the lens space case the computations displayed in section 4 seem to show an analogous situation.

It is convenient to define signs on $Rect(G)$, rather than directly on $Rect^\circ(G)$; moreover the signs will not depend on the choice of a knot, but just on the parameters of the grid.

Definition 3.1. *Given a grid diagram G , a sign assignment on G is a function*

$$\mathcal{S} : Rect(G) \longrightarrow \{\pm 1\}$$

such that the following conditions hold:

- (1) *If $r_1 * r_2 = r_3 * r_4$ then $\mathcal{S}(r_1)\mathcal{S}(r_2) = -\mathcal{S}(r_3)\mathcal{S}(r_4)$*
- (2) *If $r_1 * r_2$ is a horizontal annulus (α -strip), then $\mathcal{S}(r_1)\mathcal{S}(r_2) = 1$*
- (3) *If $r_1 * r_2$ is a vertical annulus (β -strip), then $\mathcal{S}(r_1)\mathcal{S}(r_2) = -1$*

Such a sign \mathcal{S} can be used to promote $\widehat{GC}(G)$ and $GC^-(G)$ from $\mathbb{F}[V_1, \dots, V_n]$ to $\mathbb{Z}[V_1, \dots, V_n]$ complexes.

We will prove in Theorem 3.7 that sign assignments actually exist on twisted grid diagrams, and deal with problems relating their uniqueness later on.

To see why the properties given in the previous definition are indeed

the right ones, fix a sign assignment \mathcal{S} for G , and define

$$\partial_{\mathcal{S}}(x) = \sum_{y \in S(G)} \sum_{\substack{r \in \text{Rect}^{\circ}(x,y) \\ r \cap \mathbb{X} = \emptyset}} \mathcal{S}(r) \left(\prod_{i=1}^n V_i^{O_i(r)} \right) y$$

Now we can examine the coefficient of a generator $z \neq x$ in $\partial_{\mathcal{S}}^2(x)$; each polygon connecting x to z can be decomposed in two ways (as seen in Prop 2.13). The pairs corresponding to inequivalent decompositions of the same polygon cancel out due to condition (1) on \mathcal{S} .

If instead $x = z$ there are exactly $2n$ possible ways of connecting a generator to itself with empty polygons, which are α and β degenerations; as noted before all of these strips contain one \mathbb{X} marking, so they do not contribute to the differential.

In order to prove existence, we are going to adopt the approach used in [16], which relies on the paper [6] of Gallais regarding the Spin extension of the permutation groups, introduced in the next definition.

Definition 3.2. *The Spin central extension of the symmetric group \mathfrak{S}_n is the group $\tilde{\mathfrak{S}}_n$ generated by the elements*

$$\langle z, \tilde{\tau}_{i,j} \mid 1 \leq i \neq j \leq n \rangle$$

subject to the following relations:

- $z^2 = 1$ and $z\tilde{\tau}_{i,j} = \tilde{\tau}_{i,j}z$ for $1 \leq i \neq j \leq n$
- $\tilde{\tau}_{i,j}^2 = z$ and $\tilde{\tau}_{i,j} = z\tilde{\tau}_{j,i}$
- $\tilde{\tau}_{i,j}\tilde{\tau}_{k,l} = z\tilde{\tau}_{k,l}\tilde{\tau}_{i,j}$ for distinct $1 \leq i, j, k, l \leq n$
- $\tilde{\tau}_{i,j}\tilde{\tau}_{j,k}\tilde{\tau}_{i,j} = \tilde{\tau}_{j,k}\tilde{\tau}_{i,j}\tilde{\tau}_{j,k} = \tilde{\tau}_{i,k}$ for distinct $1 \leq i, j, k \leq n$

Remark 3.3. The name Spin central extension is justified by the fact that this group can be derived as a $\mathbb{Z}/2\mathbb{Z}$ extension of \mathfrak{S}_n induced by the short exact sequence

$$(3.1) \quad 1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \tilde{\mathfrak{S}}_n \xrightarrow{\pi} \mathfrak{S}_n \longrightarrow 1$$

π is the surjective homomorphism defined by $\pi(z) = 1$ and $\pi(\tilde{\tau}_{i,j}) = \tau_{i,j}$.

Definition 3.4. *A section for $\tilde{\mathfrak{S}}_n$ is a map*

$$\rho : \mathfrak{S}_n \longrightarrow \tilde{\mathfrak{S}}_n$$

such that $\pi \circ \rho = \text{Id}_{\mathfrak{S}_n}$. We will make a slight notational abuse, and also call sections the maps

$$\rho : \mathfrak{S}_n \times (\mathbb{Z}_p)^n \longrightarrow \tilde{\mathfrak{S}}_n \times (\mathbb{Z}_p)^n$$

given by taking the product of a section with the identity map on $(\mathbb{Z}_p)^n$.

We are going to define a map

$$(3.2) \quad \varphi : \text{Rect}(G) \longrightarrow \tilde{\mathfrak{S}}_n \times (\mathbb{Z}_p)^n$$

that associates to a rectangle $r \in \text{Rect}(x, y)$ an element in $\tilde{\mathfrak{S}}_n \times (\mathbb{Z}_p)^n$, enabling us to “compare” the generators that compose the vertices of r .

If the elements of x and y in the bottom edge of r belong respectively to β_i and β_j , the first component of $\varphi(r)$ is given by the generalized transposition $\tilde{\tau}_{i,j}$. The second component of φ is given by the difference between the p -coordinates of x and y . The two generators differ only in two components, so necessarily

$$(a_1^x - a_1^y, \dots, a_n^x - a_n^y) = (0, \dots, 0, \pm k, 0, \dots, 0, \mp k, 0, \dots, 0)$$

for some $k \in \{0, \dots, p-1\}$.

Remark 3.5. To simplify the proof of the next theorem, we observe here that the generalized permutation part of the map φ does not depend on the eventual “wrapping” of a rectangle on the grid, while the $(\mathbb{Z}_p)^n$ part does.

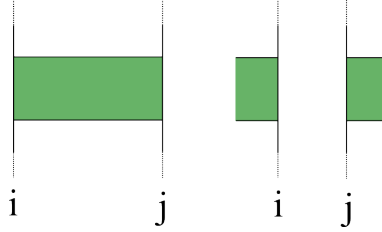


FIGURE 3.1. The generalized transpositions associated to these two rectangles are $\tilde{\tau}_{ij}$ and $\tilde{\tau}_{ji} = z\tilde{\tau}_{ij}$.

Example 3.6. Consider the rectangles R in the left part of Fig. 2.12; the value $\varphi(R)$ associated is $(\tilde{\tau}_{1,3}, (0, -1, 0, 1, 0))$ for the horizontal one and $(\tilde{\tau}_{4,5}, (0, 0, 0, 0, 0))$ for the vertical.

Given a section ρ we can build a sign assignment as follows¹²:

$$(3.3) \quad \mathcal{S}_\rho(r) = \begin{cases} 1 & \text{if } \rho(x)\varphi(r) = \rho(y) \\ -1 & \text{if } \rho(x)\varphi(r) = z\rho(y) \end{cases}$$

for $r \in \text{Rect}(x, y)$.

Theorem 3.7. *For a given section ρ , the function \mathcal{S}_ρ defined above is a sign assignment.*

Proof. First we deal with α -strips; write

$$\begin{aligned} \varphi(R_1) &= (\tilde{\tau}_{i,j}, (0, \dots, k, -k, \dots, 0)) \\ \varphi(R_2) &= (\tilde{\tau}_{j,i}, (0, \dots, -k, k, \dots, 0)) \end{aligned}$$

With $R_1 \in \text{Rect}^\circ(x, y)$, $R_2 \in \text{Rect}^\circ(y, x)$. So if

$$\rho(x)\varphi(R_1) = \rho(y)$$

¹²The operation on $\tilde{\mathfrak{S}}_n \times (\mathbb{Z}_p)^n$ considered consists in the product of permutations on the first factor, and addition on the p -coordinates.

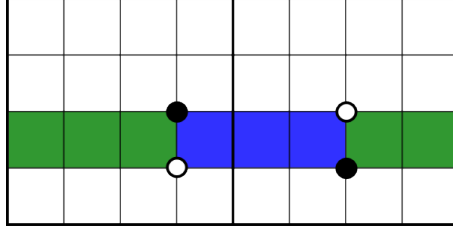


FIGURE 3.2. An α -strip of height 1. Only the relevant parts of the generators are shown.

then

$$\rho(x) = \rho(x)\varphi(R_1)\varphi(R_2) = \rho(y)\varphi(R_2)$$

which implies $\mathcal{S}(R_1) = \mathcal{S}(R_2) = 1$.

If instead we had

$$\rho(x)\varphi(R_1) = z\rho(y) \Rightarrow \mathcal{S}(R_1) = -1$$

then

$$z\rho(x) = \rho(y)\varphi(R_2) \Rightarrow \mathcal{S}(R_2) = -1$$

In both cases $\mathcal{S}(R_1)\mathcal{S}(R_2) = 1$.

Next we examine the behavior of signs for β -strips. As in the previous case there is only one other possible generator y that induces a decomposition of an annulus starting from x . The permutation components of the two rectangles $R_1 \in \text{Rect}(x, y)$ and $R_2 \in \text{Rect}(y, x)$ are both $\tilde{\tau}_{i,j}$. So if

$$\rho(x)\varphi(R_1) = \rho(y)$$

$$\rho(x)z = \rho(x)\varphi(R_1)\varphi(R_2) = \rho(y)\varphi(R_2)$$

which implies $\mathcal{S}(R_1)\mathcal{S}(R_2) = -1$.

The centrality of z tells us that the case with $\mathcal{S}(R_1) = -1$ is identical. Now, given a general polygon $P = r * r'$ connecting two generators $x \neq x'$, it is easy to check that Definition 3.2 implies

$$(3.4) \quad \rho(x') = z^{\frac{1-\mathcal{S}(r)\mathcal{S}(r')}{2}} \varphi(r)\varphi(r')\rho(x)$$

According to the proof of Prop 2.13, each polygon which is not a degeneration can be written as the concatenation of two distinct pairs of rectangles; so we just need to check for all possible polygons

$$P = r(x, y) * r(y, x') = r(x, w) * r(w, x')$$

that the following identity is verified:

$$(3.5) \quad \varphi(r(x, y))\varphi(r(y, x')) = z\varphi(r(x, w))\varphi(r(w, x'))$$

where $y \neq w$ are two auxiliary generators which differ by only one transposition from x and z . All we need to do is verify equation 3.5 on the cases $M = 2, 3, 4$ from the proof¹³ of Prop 2.13.

¹³We already considered $M = 0$, and $M = 1$ was discarded.

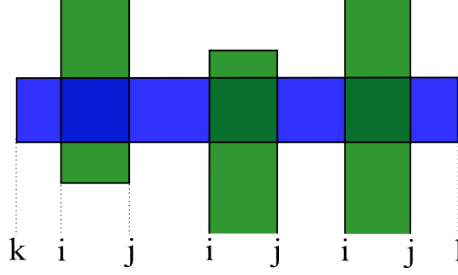


FIGURE 3.3. The generalized permutations associated are $\tilde{\tau}_{ij}$ and $\tilde{\tau}_{kl}$.

It is easy to check that the generalized permutations associated to polygons corresponding to the $M = 3$ case are the same of [16, Ch. 15] in the S^3 case; in particular this is true even when the rectangles wrap around the grid, since the generalized permutation part does not depend on the p -coordinates of the generators.

$M = 4$ is immediate: as shown in Figure 3.3 the permutations associated to the two decompositions are such that equation 3.5 becomes exactly the third relation defining $\tilde{\mathfrak{S}}_n$.

Lastly we deal with $M = 2$; the generalized transpositions associated to $r(x, y)$ and $r(y, x')$ are $\tilde{\tau}_{ij}$ and $\tilde{\tau}_{ji}$. For the two rectangles $r(x, w)$ and $r(w, x')$ on the right in Fig. 3.4 the associated transposition is $\tilde{\tau}_{ij}$ in both cases. So in particular this implies that if $\mathcal{S}(r(x, y))\mathcal{S}(r(y, x')) = -1$ then $\mathcal{S}(r(x, w))\mathcal{S}(r(w, x')) = 1$ and viceversa, and 3.5 is always satisfied. \square

Remark 3.8. It is worth noting that the trivial choice for signs (treating each rectangle just as a generalized permutation, like for the S^3 setting) can't distinguish a β degeneration from other polygons which admit two distinct decompositions into rectangles, as shown in Fig. 3.6.

Remark 3.9. The techniques used in [16, Chap.15] can be applied *verbatim* for sign assignments in lens spaces, proving that each sign assignment is induced by exactly two sections.

Now, for the uniqueness denote by $\mathcal{G}auge(G)$ the group of maps

$$v : S(G) \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

$\mathcal{G}auge(G)$ acts on sections as follows:

$$(3.6) \quad \rho^v(x) = \begin{cases} \rho(x) & \text{if } v(x) = 1 \\ z\rho(x) & \text{if } v(x) = -1 \end{cases}$$

This action is free and transitive; $\mathcal{G}auge(G)$ also acts on the set of sign assignments: if S is a sign on a grid G and $v \in \mathcal{G}auge(G)$, define $S^v(r) = v(x)S(r)v(y)$ for $r \in Rect(x, y)$.

As in the S^3 case it is easy to show that there is only one sign assignment on a grid, up to this action of $\mathcal{G}auge(G)$. The uniqueness now

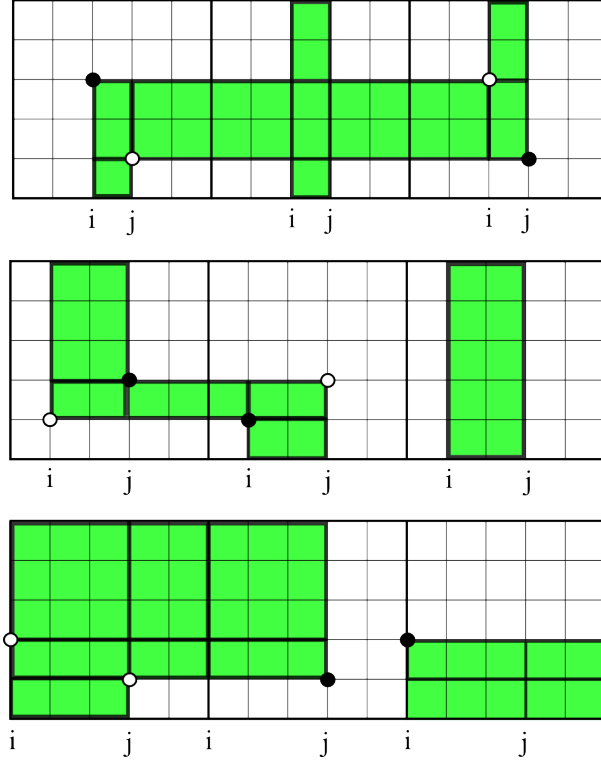


FIGURE 3.4. The generalized transpositions $\tilde{\tau}_{ij}$ and $\tilde{\tau}_{ji}$ are associated to the two decompositions in the $M = 2$ case.

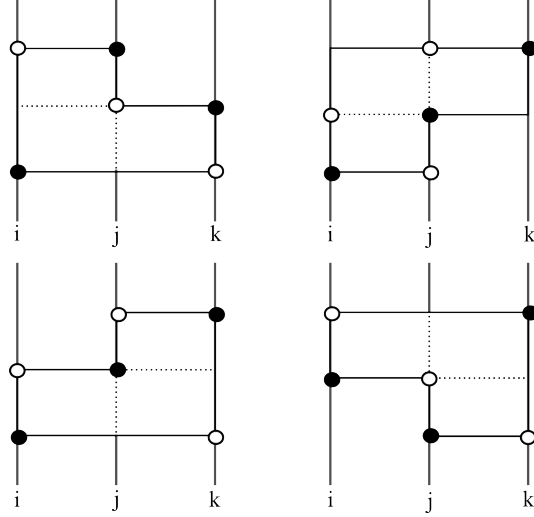


FIGURE 3.5. The four relevant combinatorial possibilities for the $M = 3$ case in the S^3 setting. Remember that the eventual wrapping of one rectangle over the other does not change the relations in $\tilde{\mathfrak{S}}_n$.

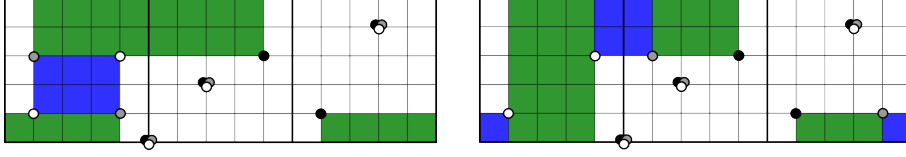


FIGURE 3.6. The white and black generators have the same permutation component, but the polygon connecting them admits two distinct decompositions. In particular it can not be an α/β -strip.

follows by noting that if S_1 and S_2 are two sign assignments on a grid G , then $S_2 = S_1^v$ for some $v \in \text{Gauge}(G)$, and the map

$$f : (GC^-(G), \partial_{S_1}) \longrightarrow (GC^-(G), \partial_{S_2})$$

given by $f(x) = v(x)x$ is an isomorphism (of trigraded R modules).

4. COMPUTATIONS

4.1. The programs: It becomes immediately apparent that the work needed to actually compute $\widehat{GH}(G)$ for grids with dimension greater than 3 is not manageable by hand¹⁴. So the author developed several programs in **SDGE** (see [4]) capable of computing the hat flavored grid homology of links in lens spaces. The computation can be made with \mathbb{Z} coefficients, provided that the grid dimension is less than 5.

By using this tool we were able to verify that all knots with a grid representative whose parameters satisfy the following conditions¹⁵, are r -torsion free ($r \leq 17$):

- for $n = 2$, $p \leq 12$
- for $n = 3$, $p \leq 6$
- for $n = 4$, $p \leq 4$
- $n = 5$, $p \leq 2$

The programs can be freely used online at my homepage:

<http://poisson.dm.unipi.it/~celoria/#programs>

Ongoing and future projects include the possibility of computations with GH^- and of the τ invariants.

4.2. Grid homology calculator. The input consists in the grid parameters (n, p, q) , followed by two strings of length n determining the positions of the \mathbb{X} and \mathbb{O} markings. We encode the markings with a string of length n for each kind; to the i -th marking (from the bottom row) we associate the number of the small square containing it (from the left, and starting from 0). As an example, the knot in Figure 2.2 is encoded as $\mathbb{X} = [12, 1, 8, 5, 9]$ and $\mathbb{O} = [6, 3, 0, 9, 12]$.

The output consists of the following:

¹⁴The generating set for a grid with parameters (n, p, q) has $n!p^n$ elements!

¹⁵These values will be updated.

- (Optional) A drawing of the chosen grid
- The hat grid homology¹⁶ $\widehat{GH}(G, \mathfrak{s}; \mathbb{Z})$ for each $\mathfrak{s} \in Spin^c$ structure, and its decategorification.
- Whether the knot is rationally fibered, the homology class and its rational genus (see [10] for the definitions).
- (Optional) A long list of the generators with their bigrading.
- (Optional) A drawing of the grid for the lift of the knot to S^3 , together with its (univariate) Alexander polynomial and the number of components of the lift.

Basically the program creates the generators $S(G)$ and computes their tridegree; afterwards it checks for empty rectangles, and creates the matrices of the differentials.

Rather than computing the module $\widehat{GC}(G)$, we adopt the simpler approach of computing yet another version of the grid homology, known as *tilde* flavored homology, $\widetilde{GH}(G)$.

The complex is simply the free \mathbb{Z} module generated over $S(G)$, and the differential counts only those empty rectangles that do not contain any marking:

$$\tilde{\partial}(x) = \sum_{y \in S(G)} \sum_{\substack{r \in Rect^{\circ}(x,y) \\ (\mathbb{X} \cup \mathbb{O}) \cap r = \emptyset}} \mathcal{S}(r)y$$

where \mathcal{S} is a sign assignment.

Using the amazing group theoretic capabilities of `sage`, the relations in $\tilde{\mathfrak{S}}_n$ (for $n \leq 5$) were encoded in a matrix associated to the differential.

The tilde flavored version is not an invariant of the knot represented by the grid. This can be easily seen *e.g.* by computing $\widetilde{GH}(G)$ in any $Spin^c$ degree, for the grids of example 2.11. However the hat version can be recovered from it as shown in the next Proposition:

Proposition 4.1 ([16]). *Given a grid G of dimension n representing the knot $K \subset L(p, q)$, there is a graded isomorphism*

$$\widetilde{GH}(G) = H_* \left(\widehat{GC}(G), \tilde{\partial} \right) \cong \widehat{GH}(L(p, q), K) \otimes W^{\otimes(n-1)}$$

where $W = \mathbb{Z}_{[0,0]} \oplus \mathbb{Z}_{[-1,-1]}$.

After computing the homology $\widetilde{GH}(G)$, the program “factors out” the tensor product dependent on the size of the grid, and prints the requested informations.

4.3. The Atlas: The speed of the previous program depends heavily on the parameters; it is painfully slow for $n \geq 6$. Another project I am currently managing is to keep a library of already computed knots;

¹⁶If the grid dimension is greater than 5 it returns the \mathbb{F} version.

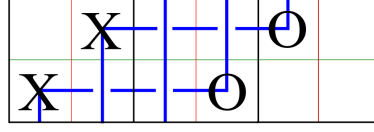


FIGURE 4.1. The knot in $L(3,1)$ described by $\mathbb{X}, \mathbb{O} = [0, 1], [3, 4]$.

this the Lens Space Knot Atlas¹⁷ *a.k.a.* **LSKA**.

Since it only has to read from existing files its speed is more or less independent from the parameters; **LSKA** encompasses all knots (links soon to come) for parameters in the following ranges¹⁸:

- $n = 1$ you can choose p up to 20.
- $n = 2$ you can choose p up to 10.
- $n = 3$ you can choose p up to 5.
- $n = 4$ you can choose p up to 2.
- $n > 4$ soon to appear.

4.4. A small example. Knot theory (and hence grid homology) in lens spaces is much more complicated than its 3-sphere counterpart: besides the fact that knots need not to be homologically trivial, they also can be nontrivial for small grid parameters. As an example, define

$$f(p) = \min\{\text{dimension of a grid representing a nontrivial knot in } L(p, q)\}$$

Then $f(1) = 5$, $f(2) = 3$ and $f(p > 2) = 2$.

Example 4.2. We sketch here the computation for the various flavors of grid homology in the case of the knot in Figure 4.1. The generating set $S(G, s)$ in each $Spin^c$ degree $s \in \{0, 1, 2\}$ has 6 elements, which we will denote x_s^0, \dots, x_s^5 for¹⁹ $s = 0, 1$. We can now list the generators, with their bidegree and differential:

¹⁷<http://poisson.dm.unipi.it/~celoria/LeSKA.html>

¹⁸These values are constantly updated.

¹⁹The homologies in $Spin^c$ degrees 1 and 2 have a similar behavior, so we omit the latter.

generator	(M, A)	differential
<hr/> <i>Spin^c</i> degree = 0 <hr/>		
x_0^0	$(\frac{3}{2}, 1)$	$\partial(x_0^0) = 0$
x_0^1	$(\frac{1}{2}, 0)$	$\partial(x_0^1) = (V_1 - V_2)x_0^0$
x_0^2	$(\frac{1}{2}, 0)$	$\partial(x_0^2) = (V_2 - V_1)x_0^0$
x_0^3	$(-\frac{1}{2}, -1)$	$\partial(x_0^3) = V_2(x_0^1 + x_0^2)$
x_0^4	$(-\frac{1}{2}, -1)$	$\partial(x_0^4) = V_1(x_0^1 + x_0^2)$
x_0^5	$(-\frac{3}{2}, -2)$	$\partial(x_0^5) = -V_1x_0^3 + V_2x_0^4$
<hr/> <i>Spin^c</i> degree = 1 <hr/>		
x_1^0	$(\frac{7}{6}, 0)$	$\partial(x_1^0) = -x_1^1 + x_1^2$
x_1^1	$(\frac{1}{6}, 0)$	$\partial(x_1^1) = 0$
x_1^2	$(\frac{1}{6}, 0)$	$\partial(x_1^2) = 0$
x_1^3	$(\frac{1}{6}, -1)$	$\partial(x_1^3) = x_1^4 - x_1^5$
x_1^4	$(-\frac{5}{6}, -1)$	$\partial(x_1^4) = V_2x_1^1 - V_1x_1^2$
x_1^5	$(-\frac{5}{6}, -1)$	$\partial(x_1^5) = V_2x_1^1 - V_1x_1^2$

Since $\widetilde{GC}(G, 0)$ has no differentials²⁰, the homology coincides with the complex. In *Spin^c* degree 1 instead the tilde homology is generated by x_1^1 and x_1^4 , so $\widetilde{GH}(G, 1) \cong \mathbb{Z}_{[\frac{1}{6}, 0]} \oplus \mathbb{Z}_{[-\frac{5}{6}, -1]}$.

The computation of the minus flavor is just slightly more involved; $GH^-(L(3, 1), K, 0)$ is composed by a copy of $\mathbb{Z}[U]$ generated by x_0^0 , plus two U -torsion components, generated by $x_0^1 + x_0^2$ and $x_0^3 + x_0^4$. Altogether

$$GH^-(L(3, 1), K, 0) = \mathbb{Z}[U]_{[\frac{3}{2}, 1]} \oplus \mathbb{Z}_{[\frac{1}{2}, 0]} \oplus \mathbb{Z}_{[-\frac{1}{2}, -1]}$$

In the last case we get

$$GH^-(L(3, 1), K, 1) = \mathbb{Z}[U]_{[\frac{1}{6}, 0]}$$

generated by x_1^1 . The hat homology can be obtained either by factoring out the tensor product with $\mathbb{Z}_{[0, 0]} \oplus \mathbb{Z}_{[-1, -1]}$ from the tilde, or deleting all dotted differentials in the minus complex of Figure 4.2, then computing the homology.

$$(4.1) \quad \widehat{GH}(L(3, 1), K, i) = \begin{cases} \mathbb{Z}_{[\frac{3}{2}, 1]} \oplus \mathbb{Z}_{[\frac{1}{2}, 0]} \oplus \mathbb{Z}_{[-\frac{1}{2}, -1]} & \text{if } i = 0 \\ \mathbb{Z}_{[\frac{1}{6}, 0]} & \text{if } i = 1 \end{cases}$$

Remark 4.3. This particular knot is interesting for several reasons: first of all it is the smallest non trivial knot; it can be also proved that despite being nullhomologous it is not even concordant to a local knot.

²⁰Its differential can be obtained by ∂ , setting all V_i variables to 0.

The methods that allow one to prove this will be detailed in a future paper.

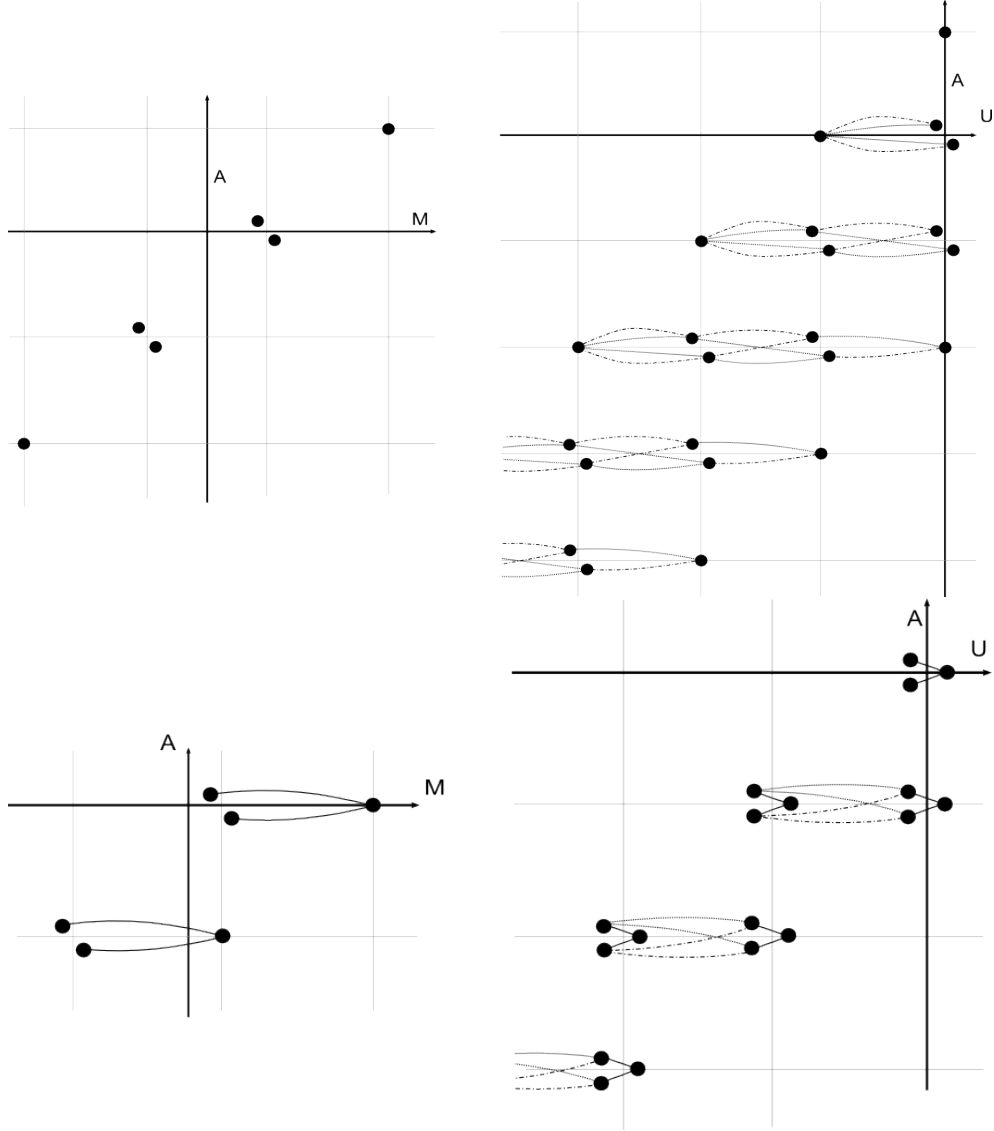


FIGURE 4.2. The complexes $\widetilde{GC}(G, i)$ (on the left) and $GC^-(G, i)$ (on the right) for $i = 0, 1$. Dotted lines denote multiplication by V_1 and dashed lines by V_2 .

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